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**LETTER TO THE EDITOR**

# Borel transform, resurgence and the density of states: lattice Schrödinger operators with exponential disorder

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**Abstract.** We study the averaged Green function of a tight-binding-model with exponentially distributed disorder as an explicit example for a resurgent function in the sense of Ecalle. As an application, one can find the (exact) asymptotic decay of the density of states and the large-order behaviour of the coefficients of the perturbation expansion.

The motion of a quantum mechanical particle in a random potential on the $\nu$-dimensional lattice $Z^\nu$ is described by the Hamiltonian

$$H = H_0 + V, \tag{1}$$

where $H_0$ has matrix elements

$$(H_0)(i,j)\begin{cases} 1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (i, j \in Z^\nu), \tag{2}$$

and the random potential $V$ consists of independent and indentically distributed random variables $V(i)$ $(i \in Z^\nu)$ with distribution $\mathrm{d}\lambda(V)$. In order to make clear the main ideas of this letter we discuss the exponential model

$$\mathrm{d}\lambda(V) = \begin{cases} 0 & \text{if } V \geqslant 0 \\ \exp(-|V|)\,\mathrm{d}V & \text{if } V < 0, \end{cases} \tag{3}$$

although other models can also be considered.

The averaged Green function

$$\overline{G(E, x, y)} = \int \mathrm{d}\lambda(V) \langle x|(H-E)^{-1}|y\rangle \tag{4}$$

can be written as a Neumann (random path) expansion for the matrix elements of the resolvent $(H-E)^{-1}$ [1]:

$$\overline{G(E, x, y)} = \sum_{\omega: x \to y} \prod_{j \in \omega} \int [(V_j - E)^{n_j(\omega)}]^{-1}\,\mathrm{d}\lambda(V_j) \tag{5}$$

Where $\omega$ is a random path on $Z^\nu$ and $n_j(\omega)$ is the number of times $\omega$ visits $j \in Z^\nu$. The series (5) can be obtained by expanding the resolvent $(H-E)^{-1}$ around its diagonal part and labelling the resulting terms by lattice paths $\omega$. It is similar in structure to a cluster (high temperature or polymer) expansion of statistical mechanics. For large

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energies $|E| \gg 1$ the expansion (5) is absolutely and uniformly convergent [1]. By expanding the factors $(V_j - E)^{-n_j(\omega)}$, one obtains a (formal) divergent expansion in powers of $E^{-1}$. It is not difficult to prove along the lines of [2] that this expansion is Borel summable in $E^{-1}$ although the density of states

$$\rho(E) = \operatorname{Im} \overline{G(E + \mathrm{i}0, 0, 0)} \tag{6}$$

does not share this property [2] ($\rho$ cannot be Borel summable because it decreases exponentially for $E \to -\infty$). Nevertheless, $\rho$ can be uniquely reconstructed from the perturbation expansion of $\bar{G}$ [2].

In this letter we will be concerned with the Borel transform $B(t)$ of $\overline{G(E)}$. We study the analytic properties of $B(t)$ in the (complex) Borel variable $t$ and prove that it has a simple 'resurgent' structure (for the definition of resurgence see below). The detailed knowledge of the analytic structure of $B$ opens some perspectives for the rigorous study of $\bar{G}$, in particular the density of states. We give some applications concerning the large-order behaviour of the coefficients of the perturbation expansion for $\bar{G}$ and the decay of the density of states for $E \to -\infty$. We mention that there is some recent interest in applying resurgence to renormalisation group ideas in dynamical systems and quantum field theory [3, 4].

Let $\hat{f}(t)$ be the Borel transform of $f(x)$ defined through the Laplace transform

$$f(x) = \int_0^\infty \hat{f}(t) \exp(-tx)\, \mathrm{d}t. \tag{7}$$

Then, if $F = f_1 \cdot f_2$, we have

$$\hat{F}(t) = (\hat{f}_1 * \hat{f}_2)t) \tag{8}$$

where the convolution $*$ is given by

$$(\hat{f}_1 * \hat{f}_2)(t) = \int_0^t \hat{f}_1(\tau)\hat{f}_2(t-\tau)\, \mathrm{d}\tau. \tag{9}$$

For $F = f_0 f_1 f_2 \ldots f_n$, (8) and (9) give

$$\hat{F}(t) = \int_0^t \mathrm{d}\tau_n \hat{f}_n(\tau_n) \int_0^{t-\tau_n} \mathrm{d}\tau_{n-1} \hat{f}_{n-1}(\tau_{n-1}) \ldots \int_0^{t-\Sigma_{i=3}^n \tau_i} \mathrm{d}\tau_2 \hat{f}_2(\tau_2)$$
$$\times \int_0^{t-\Sigma_{i=2}^n \tau_i} \mathrm{d}\tau_1 \hat{f}_1(\tau_1)\hat{f}_1\left(t - \sum_{i=1}^n \tau_i\right). \tag{10}$$

In our model, we rewrite the expansion (5) as

$$\overline{G(E, x, y)} = \sum_{\omega: x \to y} \prod_{j \in \omega} E^{-(n_j - 1)} \int_0^\infty \frac{\exp(-VE)}{(1+V)^{n_j}}\, \mathrm{d}V \tag{11}$$

which is obtained by scaling the integration variable. From (11) it is obvious that the Borel transform $B(t)$ of $\bar{G}$ is given by a sum of convolutions of powers of $t$ with integrals of the form

$$I(t) = \int_0^t \frac{\mathrm{d}\tau_m}{(1+\tau_m)^{n_m}} \int_0^{t-\tau_m} \frac{\mathrm{d}\tau_{m-1}}{(1+\tau_{m-1})^{n_{m-1}}} \cdots \int_0^{t-\Sigma_{i=3}^m \tau_i} \frac{\mathrm{d}\tau_2}{(1+\tau_2)^{n_2}}$$
$$\times \int_0^{t-\Sigma_{i=2}^m \tau_i} \frac{\mathrm{d}\tau_1}{(1+\tau_1)^{n_1}} \left[\left(1 + t - \sum_{i=1}^m \tau_i\right)^{n_0}\right]^{-1}, \tag{12}$$

where we have denoted the number of sites in $\omega$ by $|\omega|=m+1$, $m\geqslant 1$. The powers of $t$ present no difficulties such that we resume to study the typical integral $I(t)$. We use the Feynman formula [5]

$$\frac{1}{\Pi_{j=1}^{n}A_j^{\lambda_j}}=\frac{\Gamma(\lambda_0)}{\Pi_{j=1}^{n}\Gamma(\lambda_j)}\int_0^1\prod_{j=1}^{n}\mathrm{d}\alpha_j\delta\left(1-\sum_{j=1}^{n}\alpha_j\right)\frac{\Pi_{j=1}^{n}\alpha_j^{\lambda_j-1}}{(\Sigma\,\alpha_jA_j)^{\lambda_0}} \tag{13}$$

where$\lambda_0=\Sigma_{j=1}^{n}\lambda_j$. This gives

$$\begin{aligned}I(t)=\frac{\Gamma(\Sigma_{i=0}^{m}n_i)}{\Pi_{i=0}^{m}\Gamma(n_i)}&\int_0^1\prod_{j=0}^{m}\mathrm{d}\alpha_j\,\delta\left(1-\sum_{j=0}^{m}\alpha_j\right)\prod_{j=0}^{m}\alpha_j^{n_j-1}\\&\times\int_0^t\mathrm{d}\tau_m\int_0^{t-\tau_m}\mathrm{d}\tau_{m-1}\ldots\int_0^{t-\Sigma_{i=3}^{m}\tau_i}\mathrm{d}\tau_2\int_0^{t-\Sigma_{i=2}^{m}\tau_i}\mathrm{d}\tau_1\\&\times\left(1+\alpha_0t+\sum_{i=1}^{m}\tau_i(\alpha_i-\alpha_0)\right)^{-\Sigma_{i=0}^{m}n_i}.\end{aligned} \tag{14}$$

The $\tau$ integrals can be computed explicitly. In order to study the analytic structure of $I(t)$ we restrict ourselves to $n_i=1$, $i=0,1,\ldots,m$; the general case is similar (in fact through partial integrations in (11) we can reduce the general case to this particular one). After doing the $\tau$ integrals we get (up to the combinatoric constants)

$$I(t)\simeq\int_0^1\prod_{j=0}^{m}\mathrm{d}\alpha_j\,\delta\left(1-\sum_{j=0}^{m}\alpha_j\right)t^m\left(\prod_{j=0}^{m}(1+\alpha_jt)\right)^{-1}. \tag{15}$$

The advantage of (15) over (14) is that all integrals are now taken between constant limits. We can now study the analytic structure of $I(t)$ in the complex $t$ plane by using the powerful theory of Landau singularities for Feynman integrals [5]. We find that it has a particularly simple singularity structure: there are only isolated (simple poles) and logarithmic singularities present at $t=-1,-2,\ldots$. This reminds us of the so-called resurgent structure [6]. A function is called resurgent if its Borel transform $B(t)$ has only isolated and logarithmic singularities, i.e. has the form locally of

$$B(t)=B_\tau^{(1)}(t-\tau)+[B_\tau^{(2)}(t-\tau)]/(t-\tau)+\log(t-\tau)B_\tau^{(3)}(t-\tau) \tag{16}$$

for $t$ near $\tau$, where $B_\tau^{(1,2,3)}$ are regular near the origin. Resurgent functions have recently been encountered in several areas of mathematical physics including the renormalisation group approach to dynamical systems and quantum field theory [3, 4, 7, 8]. It is not difficult to prove by looking on combinatoric factors that the random path expansion (11) is convergent in the Borel variable. We deduce that the averaged Green function of the exponential model has the resurgent structure. It must be remarked that this is a type of 'instanton' resurgence and it may be qualitatively different from the 'renormalon' resurgence [7].

Finally we remark that studying (cluster or polymer) expansions of type (11) in the Borel variable may have some advantages. As an example [9] we have extracted from the nearest singularity of $B(t)$ to the origin (in the exponential model it sits at $t=-1$) the exact decay (including pre-exponential factors) of the density of states for $E\to-\infty$ and the large-order behaviour of the perturbation expansion for the averaged Green function.

The connections between decay rate of the density of states and the large order behaviour of perturbation theory are discussed in references [10–12]. We consider our result as a rigorous justification of the Brézin–Parisi formula (equation (4) of [12]) for our model.

We remark that other models can be studied analogously [9].

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